

# Interacting Constituents in Cosmology

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## Abstract

Universe evolution, as described by Friedmann's equations, is determined by source terms fixed by the choice of pressure  $\times$  energy-density equations of state  $p(\rho)$ . The usual approach in Cosmology considers equations of state accounting only for kinematic terms, ignoring the contribution from the interactions between the particles constituting the source fluid. In this work the importance of these neglected terms is emphasized. A systematic method, based on the Statistical Mechanics of real fluids, is proposed to include them. A toy-model is presented which shows how such interaction terms can engender significant cosmological effects.

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## 1 Introduction

The greatest achievement of Physical Cosmology is, up to now, the Big Bang standard model. This model is based on two principles which are consistent with large-scale observations: the *cosmological principle* and the *universal time principle*. The first postulates that the universe space-section  $\Gamma$  is homogeneous and isotropic; the latter states that the topology of the space-time four-dimensional manifold  $\mathcal{M}$  is the direct product  $\mathcal{M} = \mathbb{R} \times \Gamma$ , and results from the adoption of a cosmological time as parameter of the manifold foliation [1, 2]. These principles lead [3, 4] to the Friedmann-Lemaître-Robertson-Walker (FLRW) interval<sup>2</sup>

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (k = 0, \pm 1) . \quad (1)$$

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<sup>2</sup> We shall be using natural units,  $c = 1, \hbar = 1$ .

On the other hand, the source terms are described by the perfect fluid energy-momentum tensor. This tensor  $T_{\mu\nu}$  and the above interval, when substituted into the Einstein's equations of General Relativity

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} , \quad (2)$$

(with a cosmological-constant  $\Lambda$ -term) lead to the Friedmann equations for the scale factor  $a(t)$ :

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2} , \quad (3)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p) . \quad (4)$$

The solution for  $a(t)$  is obtained after inserting in these equations the constitutive relation between the pressure  $p$  and the energy-density  $\rho$  of the cosmic source fluid.

The relation between  $p$  and  $\rho$  is given directly by an *equation of state* (EOS) of type  $p = p(\rho)$ , or indirectly via the *distribution functions*. Either way, here are implicitly in use the methods of Statistical Mechanics (SM), which can, for pedagogical purposes, be divided into two branches [5]: equilibrium SM, with its ensemble technique and its partition functions; and non-equilibrium SM, whose distribution function entail, for example, kinetic equations as the Boltzmann and the Vlasov-Landau equations. The distribution function are not only useful to describe systems out of equilibrium, but also systems in thermodynamical equilibrium, to which a temperature can be attributed. The ensemble approach, however, can only be applied to systems in equilibrium.

Cosmology uses both sections of Statistical Mechanics, e.g.: the EOS obtained from a partition function leads to the description of the radiation-dominated era in the thermal history of the universe and leads also to the  $\Lambda$ CDM model for the present-day cosmic dynamics [6, 7]; the distribution function determined from Boltzmann equations are used in the perturbative cosmological models describing the cosmic microwave background anisotropies, as well as the formation of structures [8]. But the fact is that, perturbative or not, **the present formulation of Cosmology does not take into account the interactions between the constituents as direct sources of gravitation**. Only kinematical terms are computed in the ordinary (non-perturbed) cosmological models, interactions being just used to explain pair-production and thermalization; the dynamical terms – which could be introduced via particle-to-particle potentials or through the S-matrix – are simply forgotten. Even the perturbative approach [8] considers the interaction so insufficiently that the pressure function appearing in the expression of the energy-momentum tensor presents kinematical terms solely [9].

In section 2, we shall make explicit the absence of dynamical terms (interaction terms) both in the conventional cosmological models developed under the hypothesis of thermodynamical equilibrium – with their typical EOS – and in

the perturbative models (out of equilibrium) – built on the distribution functions. Based on the Statistical Mechanics of systems with interactions [5, 10] we discuss, in section 3, how to include those terms in Cosmology. Section 4 presents a toy-model which shows the decisive influence that interaction terms can have on cosmic evolution. The conclusions are summed up in section 5.

## 2 Absence of interaction terms

This section deals, in the context of the standard model, with an ideal source cosmic fluid, interactions between components being supposed absent. We leave to the next section the discussion about the real need for the inclusion of interaction terms in some phases of the universe evolution.

First, it is important to establish the period during which equilibrium Statistical Mechanics is applicable. This is restricted by the fact that the universe, described by the FRW metric (1), is growing up — the volume containing the cosmic fluid is expanding. In the primeval universe ( $kT > 20$  MeV, where  $k$  is the Boltzmann constant), the typical reaction rates  $\Gamma_{pri}$  involving the different constituents are much larger than the expansion rate  $H_{pri}$  [11], i.e.

$$\Gamma_{pri} \gg H_{pri} \equiv \frac{\dot{a}_{pri}}{a_{pri}}. \quad (5)$$

For the energy values then prevalent the thermodynamic notion of *quasi-static expansion* holds: for each infinitesimal variation of volume, all constituents of the fluid are at the same temperature — they keep themselves in thermal equilibrium. Notice in advance that, as the curvature is negligible ( $k = 0$ )<sup>3</sup> in the early universe, the ordinary Statistical Mechanics defined on euclidean 3-space  $\mathbf{E}^3$  can be used. We emphasize that Eq.(5) is valid even during an accelerated expansion.

The components of the cosmic fluid will decouple progressively as the temperature decreases [11, 8]. A natural question turns up: once decoupled from the other fluid components, will a given particular component remain in thermal equilibrium ? This question is better formulated in terms of the component distribution function: is there an equilibrium distribution function for the decoupled component valid in general in an expanding plane space ? The answer is no [12]. The proof of this statement is not trivial, but it is related to the fact that we do not have spatially constant time-like Killing vectors in plane FRW space-times [9].

Nevertheless, it is possible to construct distribution functions for the expanding space in particular cases, as in the presence of non-relativistic and ultrarelativistic components. This is comforting enough since, from the early to the present-day universe, the decoupling stable particles are either ultrarelativistic – photons and neutrinos – or non-relativistic (baryonic and dark matter).<sup>4</sup> We are therefore allowed to use equilibrium SM when working with this

<sup>3</sup> In agreement with the recent observational data, we will assume  $k = 0$  all along the text.

<sup>4</sup> We adopt cold (non-relativistic) dark matter from the beginning.

content. The non-equilibrium treatment will be required only for perturbative approximations.

Next sub-section deals specifically with the description given by the cosmological models in thermodynamical equilibrium, and 2.2 discuss the models with components out of equilibrium.

## 2.1 Cosmology for systems in equilibrium

The classical Cosmology text-books, e.g. [1, 6, 8, 11], teach us how to determine the pressure  $p_i$ , the energy density  $\rho_i$  and the numerical density  $n_i$  of the  $i$ -th component of the fluid in thermal equilibrium:

$$n_i = \frac{g_i}{(2\pi)^3} \int f_i(\vec{p}) d^3p, \quad (6a)$$

$$\rho_i = \frac{g_i}{(2\pi)^3} \int E_i(\vec{p}) f_i(\vec{p}) d^3p, \quad (6b)$$

$$p_i = \frac{g_i}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E_i(\vec{p})} f_i(\vec{p}) d^3p, \quad (6c)$$

where  $g_i$  is the degeneracy degree;  $E_i$ , the dispersion relation

$$E_i^2 = \vec{p}^2 + m_i^2; \quad (7)$$

and  $f_i(\vec{p})$  the distribution functions given by

$$f_i(\vec{p}) = \frac{1}{e^{\beta(E_i - \mu_i)} \pm 1}, \quad \beta \equiv \frac{1}{kT}. \quad (8)$$

The lower (upper) sign refers to the Fermi-Dirac (Bose-Einstein) statistics.

The sum over all the  $i$  components is taken into the Friedmann equations (3, 4). Standard texts on Statistical Mechanics [10] derive  $p_i$ ,  $\rho_i$  and  $n_i$  in the ensemble formalism for *ideal* relativistic quantum particles. The grand-canonical partition function  $\Xi_i$  or the potential  $\Omega_i$  for the  $i$ -th component are given as

$$\begin{aligned} \Omega_i(V, \beta, \mu_i) &\equiv \frac{\ln \Xi_i(V, \beta, \mu)}{V} = \pm \frac{g_i}{(2\pi)^3} \int \ln [1 \pm z_i e^{-\beta E_i}] d^3p = \\ &= \frac{g}{(2\pi)^3} \sum_{j=1}^{\infty} \frac{(\mp 1)^{j-1}}{j} \int z_i e^{-j\beta E_i} d^3p, \end{aligned} \quad (9)$$

where

$$z_i = e^{\beta \mu_i} = e^{\beta(\mu_i^{NR} + m_i)} \quad (10)$$

is the fugacity. In (10), the chemical potential  $\mu_i = \mu_i^{NR} + m_i$  has a non-relativistic contribution  $\mu_i^{NR}$ , which is the usual term appearing in text-books.

We have above included also the term  $m_i$  associated to the rest-energy of the particle under consideration. The quantities  $p_i$ ,  $\rho_i$  and  $n_i$  are, then,

$$p_i = \frac{1}{\beta} \Omega_i ; \quad (11a)$$

$$n_i = z_i \left. \frac{\partial \Omega_i}{\partial z_i} \right|_{V, \beta} ; \quad (11b)$$

$$\rho_i = - \left. \frac{\partial \Omega_i}{\partial \beta} \right|_{V, z_i} . \quad (11c)$$

These prescriptions are equivalent to the definitions (6a), (6b) and (6c).

Analyzing the ultra-relativistic limit,  $kT \gg m$ , in Eqs.(6b, 6c) and assuming  $kT \gg \mu$  (as is the case for the photons), we obtain

$$p_\gamma = \frac{\rho_\gamma}{3} , \quad (12)$$

the familiar equation of state for radiation.

Conversely, if we take the non-relativistic limit  $kT \ll m$  in the same Eqs.(6b, 6c) and neglect quantum effects, it results

$$\rho_{NR} = n_{NR} m , \quad (13a)$$

$$p_{NR} = n_{NR} kT . \quad (13b)$$

The quantum effects will be negligible if the condition  $n\lambda^3 \ll 1$  is satisfied,  $\lambda$  being the thermal wavelength

$$\lambda = \sqrt{\frac{2\pi}{mkT}} . \quad (14)$$

This will hold when no particle invades any other's effective volume, of which a rough estimate is  $\lambda^3$ : this is the meaning of the condition above.

Pressure  $p$  only appears in the Friedmann equations added to  $\rho$ . If we consider  $kT \ll m$ , Eqs.(13a, 13b) will say that

$$p_M = 0 \quad (15)$$

is a very good approximation for the equation of state for non-relativistic matter  $M$  (the dust approximation).

The Friedmann equations, together with (12) and (15), enable us to calculate simplified, but analytic, solutions for the universe evolution determined by  $a(t)$  [7]. Present-day universe, for example, is well described by the solution obtained after inserting EOS (15) in (3, 4), keeping  $\Lambda$  non-null, and neglecting the contribution of radiation — the so-called  $\Lambda$ CDM model. For the early universe, period during which radiation dominates, the suitable EOS is (12).

Anyway, neither the EOS simpler forms (12, 15) nor the complete expressions, Eqs.(11a, 11b, 11c), take interactions between the constituents into account. Partition function (9) includes all the possible states of the  $i$ -th component in an *ideal* gas, of *non-interacting* fermionic (or bosonic) relativistic

particles. The first term of series (9) refers to the classical description (Boltzmann statistics), while the others are order-by-order quantum corrections. We shall see in section 3 that the expression of the partition function in terms of a series suggests a mechanism of inclusion of the interactions order by order.

We remind the reader that our interest here is to analyze EOS based on first principles; Eqs.(12) and (13b) are examples of this class. We shall exclude any EOS used in the context of Cosmology constructed on phenomenological basis, as EOS for scalar fields [13, 14, 15, 16, 17], EOS for the Chaplygin gas [18, 19, 20, 21, 22] and Van der Walls' EOS [23], to mention but a few.

## 2.2 Cosmology for systems out of equilibrium

As said before, the use of non-equilibrium Statistical Mechanics is essentially required only in perturbative cosmology. According to [8, 9], perturbations on the FRW metric come from interactions between the components of the cosmic fluid as described by the Boltzmann equation,

$$\frac{df_i}{dt} = \sum_j C_{ij}[f_i] , \quad (16)$$

where  $f_i$  is the distribution function of the  $i$ -th component and  $C_{ij}[f_i]$  is the collision term for the  $(i, j)$  pair of components. It is written as

$$C_{ij}[f_i(\vec{p}_1)] = \sum_{\vec{p}_2, \vec{p}_3, \vec{p}_4} |M_{ij}|^2 [f_i(\vec{p}_3)f_j(\vec{p}_4) - f_i(\vec{p}_1)f_j(\vec{p}_2)] , \quad (17)$$

where  $M_{ij}$  is the scattering amplitude of two interacting particles with incoming momenta  $\vec{p}_1$  and  $\vec{p}_2$  and out-coming momenta  $\vec{p}_3$  and  $\vec{p}_4$ .

Once the distribution function is evaluated (via Boltzmann equation) for all components of the fluid, the energy-momentum tensor is obtained as

$$T^{\mu\nu} = \sum_i g_i \int f_i(\vec{p}) \frac{p^\mu p^\nu}{p_0} \frac{d^3 p}{(2\pi)^3} , \quad (18)$$

where  $p^\mu$  is the comoving momentum encapsulating, of course, the energy and the tri-momentum  $\vec{p}$ ,

$$p_0 = E ; \quad p^2 = g_{ij} p^i p^j . \quad (19)$$

As sources in Einstein's equations,  $T^{00}$  is the energy density (6b) and  $T^{ii}$  is the pressure (6c). At first sight one could think that the Boltzmann equation (16) introduces interaction terms as direct sources of gravitation, but in fact the usual pressure expression only partially considers dynamical (interaction) effects. Interactions are taken into account only indirectly, through deformations in the distribution functions  $f_i$  and  $f_j$ . Nevertheless, the distributions used are those of free particles. An example of dynamical pressure appears in the van der Walls equation

$$p = \frac{nkT}{(1 - An)} - B n^2 , \quad (20)$$

in which  $A$  and  $B$  [9] are constants phenomenologically fitted for each gas. It cannot be obtained from (18).

Let us see in more detail what happens. The deduction of the Boltzmann equation uses the fact that the number of pairs of particles with velocities  $\vec{v}_i$  and  $\vec{v}_j$  at time  $t$  is

$$f_i(\vec{x}_i, \vec{v}_i) f_j(\vec{x}_j, \vec{v}_j) . \quad (21)$$

On the other hand, Statistical Mechanics tells us that this is true only when the correlations between particles are negligible. In fact, the number of pairs of particles with velocities  $\vec{v}_i$  and  $\vec{v}_j$  is determined by the two-particle distribution function

$$f_{ij}(\vec{x}_i, \vec{x}_j, \vec{v}_i, \vec{v}_j) \quad (22)$$

and, in general,

$$f_{ij}(\vec{x}_i, \vec{x}_j, \vec{v}_i, \vec{v}_j) \neq f_i(\vec{x}_i, \vec{v}_i) f_j(\vec{x}_j, \vec{v}_j) . \quad (23)$$

The  $N$ -particle distribution function  $f_N(x_1, \dots, x_N)$  may be expressed as [5]

$$f_N(\vec{x}_1, \dots, \vec{x}_N) = \prod_i^N f_i(\vec{x}_i) + \bar{g}_N(\vec{x}_1, \dots, \vec{x}_N) ; \quad (24)$$

$\bar{g}_N$  determines the complete correlation degree of the system. For practical reasons, it is convenient to split the set of  $N$  particles in all the possible disjoint subsets containing at least one particle, i.e.

$$\begin{aligned} f_i(\vec{x}_i) &= f_i(\vec{x}_i) , \\ f_{ij}(\vec{x}_i, \vec{x}_j) &= f_i(\vec{x}_i) f_j(\vec{x}_j) + g_{ij}(\vec{x}_i, \vec{x}_j) , \\ f_{ijk}(\vec{x}_i, \vec{x}_j, \vec{x}_k) &= f_i(\vec{x}_i) f_j(\vec{x}_j) f_k(\vec{x}_k) + f_i(\vec{x}_i) g_{jk}(\vec{x}_j, \vec{x}_k) + \\ &\quad + f_j(\vec{x}_j) g_{ik}(\vec{x}_i, \vec{x}_k) + f_k(\vec{x}_k) g_{ij}(\vec{x}_i, \vec{x}_j) + g_{ijk}(\vec{x}_i, \vec{x}_j, \vec{x}_k) , \end{aligned} \quad (25)$$

etc. In these equations  $f_i$  is the one-particle distribution function;  $f_{ij}$  is the two-particle distribution function, and so on.

This is the *cluster* expansion formalism, function  $g_s(\vec{x}_1, \dots, \vec{x}_s)$  being the irreducible correlation function of  $s$  particles. These are the functions describing interactions. For example, the van der Waals equation can be obtained from the two-particle correlation  $g_{ij}(\vec{x}_i, \vec{x}_j)$  and a couple of suitable approximations.

Notice that, in spite of  $f_i$  being always positive, the function  $f_{i_1, \dots, i_N}$  is not necessarily so. This fact is exemplified by Eq.(20), where the relative values of  $A$  and  $B$  determines if the pressure is positive or negative.<sup>5</sup>

A remark: the attentive reader will have noticed that in the passage from (23) to (24) the dependence on  $\vec{v}_i$  has been suppressed. That is because the correlation functions usually depends only on the position. The  $f_N$  depend both on position and velocity, but the sector correspondent to the velocities may be separated, resulting in non-correlated distribution functions  $f_i(\vec{v}_i)$ . Figure 1 shows schematically the correspondence between the distribution functions and the clusters for a three-particle system.

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<sup>5</sup> We are not considering phase-transitions effects.

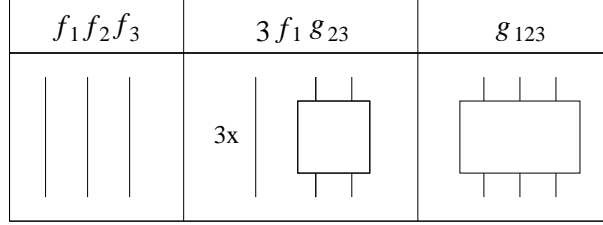


Figure 1: Cluster schematic representation for a three-particle system.

The distribution functions for a  $N$ -particle system are determined, order by order, by a hierarchical set of equations known as BBGKY – after Bogoliubov, Born, Green, Kirkwood and Yvon [24]. The Boltzmann equation is itself an approximation of the BBGKY system reducing this set of linear coupled equations to just one non-linear equation. Considering interaction in this equation will solely modify the one-particle distribution functions  $f_i$ . The statistical systems modeled by Boltzmann equation do not exhibit correlation terms in the distribution functions. Therefore, the thermodynamical quantities, like pressure, do not present dynamical terms. This leads to the conclusion that the usual approach of perturbative cosmology does not compute interaction processes as direct sources of gravitation.

We shall below discuss in which periods of cosmic history interaction should effectively contribute to gravitation, and how to include it.

### 3 On the inclusion of interaction terms

Qualitatively, the cosmic fluid can be separated into three different sectors: a kinetic part  $T$ , associated to velocities; a dynamical component  $V$ , related to interactions; and a massive contribution  $M$  coming from the rest mass of the constituents. The condition for the interaction being a relevant source is

$$V(\Delta t) \geq T(\Delta t) + M(\Delta t), \quad (26)$$

where  $\Delta t$  indicates a given cosmological period.

As far as interactions are concerned, each period presents its own characteristics. We will deal with two of them: the pre-nucleosynthesis cosmological universe (PNU) –  $kT \gtrsim 20$  MeV or red-shift  $z \gtrsim 10^{12}$  – and the recent universe – red-shift  $z \lesssim 20$ .

In the PNU, the high temperature of the fluid warrants the existence of a large variety of particles, such as  $\gamma, \nu, \pi, K$ , most of them interacting mutually.<sup>6</sup> The answer to the question about the relevance of interactions for the primeval cosmology comes from an involved analysis of the interacting fluid as a whole:

<sup>6</sup> The magnitude and type of interaction depends, of course, on the species considered.



the interaction terms may be positive or negative, depending on the nature of the interaction, and global cancellation may possibly turn up.

Recent data from RHIC (*Relativistic Heavy Ion Collider*) [25] indicates that a fluid at very high temperatures (few hundreds of MeV) presents a strong interaction between its constituents, even possibly generating a “liquid” state for the hadronic matter, the CGC (*Color Glass Condensate*) [26]. Quantum Chromodynamics suggests that this system is constituted basically by three quarks ( $u, d, s$ ) and gluons. In this energy range – which corresponds to the PNU – the strong interaction between quarks (generically,  $q$ ) overcome their kinetic and rest energies. We have, so,  $V_q(\Delta t_{PNU}) \geq T_q(\Delta t_{PNU}) + M_q(\Delta t_{PNU})$ . Even then, the primordial cosmic fluid has others species that should be considered when applying criterion (26). In section 4 we exhibit a very simplified model, introducing interaction terms, to describe the pre-nucleosynthesis cosmological universe.

Among the various components of the recent universe, one of the most important for its evolution is non-relativistic matter. Indeed, at the present-day time  $t_0$  (red-shift  $z = 0$ ), the rest mass of this component  $M_{NR}$  corresponds to about 30% of the universes’ total content [27], and, as we go back in time, it becomes more and more relevant [28]. Non-relativistic matter responds to the Newtonian gravitational interaction  $V_{NR}$  responsible for structure formation and evolution, and the experimental data indicates the influence of the potential is more effective as  $z$  diminishes (i.e., the structures grow) [29]. As the kinetic term  $T_{NR}$  is negligible compared to  $M_{NR}$ , the importance of the gravitational interaction  $V_{NR}$  is measured by its direct comparison with  $M_{NR}$ . The Newtonian potential is a long range interaction and one could ask whether it can decisively contribute with dynamical terms that could influence the present day cosmic behavior. We will not study this subject in the present work, but only mention three effects controlling its importance:

1. As the scale factor  $a(t)$  grows, so does the interaction distance  $d = a(t)r$  (where  $r$  is the comoving distance), reducing the cosmological contribution of  $V_{NR}$ .
2. During a decelerated (accelerated) expansion, the comoving horizon increases (decreases) consequently increasing (decreasing) the global effects of  $V_{NR}$ .
3. The basic constituents of the universe are different at each phase of the universe evolution, starting with fundamental particles (nucleons, etc.), passing to hydrogen clouds and then to galaxies and clusters.

Let us return to the analysis of the pre-nucleosynthesis cosmological period. Next section presents a prescription to include interaction in that period.

### 3.1 Equation of state with interaction terms

The inclusion of interactions via EOS can be done by the ensemble formalism through the perturbative treatment of real gases. Mayer and collaborators devel-

oped in 1937 the systematic approach of cluster expansion for a non-relativistic classical (non-quantum) system [30]. Just after that, in 1938, Kahn and Uhlenbeck began the generalization of this method to non-relativistic quantum statistics [31, 32], and, in 1960, Lee and Yang improved the treatment to describe, in principle, all the perturbation orders [33]. Finally, in 1969, Dashen, Ma and Bernstein extended this method to a relativistic quantum system where the interactions are computed through the  $S$  matrix [34]. Each one of these treatments apply to a different statistical system, but all of them were constructed so as to be valid on a plane static space-time. This prevents their straight application to cosmology, where one needs to consider the possibility of a curved manifold. This means that the perturbation methods for modeling real gases are valid only if the space-section of the universe is plane, and the expansion is of the quasi-static type. And these requirements, as said in section 2, are fulfilled in the pre-nucleosynthesis universe ( $kT \gtrsim 20 \text{ MeV}$ ).

According to the cluster expansion technique, the grand canonical potential for a one-component fluid is

$$\Omega(z, T) = \sum_{N=1}^{\infty} b_N z^N = \sum_{N=1}^{\infty} b_N e^{N\beta(\mu_{NR} + m)}. \quad (27)$$

The  $b_N$  are the *cluster integrals* and encapsulate all the information about the interaction processes. The Appendix is a *resumé* on cluster expansions, with the differences between classical and quantum systems particularly emphasized.

The first cluster integrals for the non-relativistic classical system are

$$b_1 = g \frac{e^{-\beta m}}{\lambda^3 V} \int d^3 r_1 = g \frac{e^{-\beta m}}{\lambda^3}, \quad (28a)$$

$$b_2 = g \frac{e^{-2\beta m}}{2\lambda^6 V} \int \int f(\vec{r}_1, \vec{r}_2) d^3 r_1 d^3 r_2, \quad (28b)$$

$$b_3 = g \frac{e^{-3\beta m}}{6\lambda^9 V} \int \int \int [f(\vec{r}_1, \vec{r}_2)f(\vec{r}_1, \vec{r}_3)f(\vec{r}_2, \vec{r}_3) + f(\vec{r}_1, \vec{r}_2)f(\vec{r}_1, \vec{r}_3) + f(\vec{r}_1, \vec{r}_2)f(\vec{r}_2, \vec{r}_3) + f(\vec{r}_1, \vec{r}_3)f(\vec{r}_2, \vec{r}_3)] d^3 r_1 d^3 r_2 d^3 r_3, \quad (28c)$$

where  $\lambda$  is the thermal wavelength,  $g$  is the degeneracy degree and

$$f(\vec{r}_i, \vec{r}_j) \equiv e^{-\beta V(\vec{r}_i, \vec{r}_j)} - 1, \quad (29)$$

are the Mayer functions. Notice that in the classical case the interaction is introduced through the interparticle potential  $V(\vec{r}_i, \vec{r}_j)$ . Though the system is nonrelativistic, the rest mass is already included in the  $b_N$  through the factors  $e^{-\beta m}$ . These are not considered by the traditional texts on non-relativistic Statistical Mechanics, but they are necessary in the derivation of coherent cosmological energy density. The pressure and the numerical density are not affected by these factors.

Dashen, Ma and Bernstein [34] have shown that the general form of the coefficients  $b_N$  for a RQS is given in terms of the S-matrix as

$$b_N - b_N^{(0)} = \frac{g}{V} \int \frac{e^{-\beta E}}{4\pi i} \text{Tr} \left( \hat{A} \hat{S}^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} \hat{S} \right)_{c_N} dE, \quad (30)$$

where  $b_N^{(0)}$  is the cluster integral of the non-interacting quantum theory,  $\hat{A}$  is the symmetrization operator,  $\hat{S}$  is the on-shell S-matrix operator [35], and  $c_N$  stands for all the  $N$ -particle connected diagrams. We are using the definition

$$\hat{S}^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} \hat{S} = \hat{S}^{-1} \frac{\partial \hat{S}}{\partial E} - \frac{\partial \hat{S}^{-1}}{\partial E} \hat{S}. \quad (31)$$

See the Appendix for further details.

Once the grand canonical potential (27) is obtained, it is straightforward to calculate the pressure (11a), the numerical density (11b) and the energy density (11c) as functions of the temperature and the fugacity:

$$p(z, kT) = kT \sum_{N=1}^{\infty} b_N z^N, \quad (32a)$$

$$n(z, kT) = \sum_{N=1}^{\infty} N b_N z^N, \quad (32b)$$

$$\rho(z, kT) = (kT)^2 \sum_{N=1}^{\infty} \frac{\partial b_N}{\partial (kT)} z^N. \quad (32c)$$

These equations are the one-component fluid EOS in parametric form.

An alternative description is obtained if everything is rewritten in terms of the numerical density after inversion of the  $n(z, T)$  series. The result is the *virial expansion*,

$$p(n, kT) = kT \sum_{l=1}^{\infty} a_l(kT) n^l, \quad (33a)$$

$$\rho(n, kT) = (kT)^2 \sum_{l=1}^{\infty} c_l(kT) n^l, \quad (33b)$$

where  $a_l$  and  $c_l$  are, respectively, the virial coefficients for the pressure and the energy density. They are completely determined by the  $b_N$ . For instance, the first three terms are:

$$a_1 = 1, \quad a_2 = -\frac{b_2}{b_1^2}, \quad a_3 = \frac{2}{b_1^3} \left( 2\frac{b_2^2}{b_1} - b_3 \right); \quad (34)$$

$$c_1 = \frac{1}{b_1} \frac{\partial b_1}{\partial (kT)} = \frac{1}{kT} \left( \frac{m}{kT} + \frac{3}{2} \right), \quad c_2 = -\frac{\partial a_2}{\partial (kT)}, \quad c_3 = -\frac{1}{2} \frac{\partial a_3}{\partial (kT)}. \quad (35)$$

The question asking for an answer now is: what is the most suitable set of EOS for Cosmology, Eqs.(32a, 32b, 32c) or (33a, 33b)? The following comments will help to determine the preferred set.

The great advantage of writing the EOS in form of series is the possibility of approximating them by their first terms. Whenever series are involved, something must be said about their *convergence*. The sets of EOS (32a, 32b, 32c) and (33a, 33b) are equivalent when all the terms of their series are considered; but that equivalence ceases to exist when the series are truncated at a given order. The convergence of the sets  $\{p(z, T), \rho(z, T)\}$  and  $\{p(n, T), \rho(n, T)\}$  depends critically on the interaction processes at play and, in consequence, on the cosmological period under consideration.

All the relevant particles for the dynamics of the PNU are born through pair-production at high temperatures (from one tenth to hundreds of MeV). Each of the species will have [1, 6] a nearly vanishing (total) chemical potential  $\mu \simeq 0$  and, therefore, a fugacity close to one,  $z \simeq 1$ . Consequently,  $z$  is not a good expansion parameter for a series.

On the other hand, the expansions of  $p(n, T)$  and  $\rho(n, T)$  describe with great accuracy rarefied non-relativistic gases. Indeed, using the Lennard-Jones potential, the correction

$$a_2 = -\frac{2\pi}{g} \int_0^\infty (e^{-V(r_{12})/kT} - 1) r_{12}^2 dr_{12} , \quad r_{12} = |\vec{r}_2 - \vec{r}_1| , \quad (36)$$

leads to an EOS that fits the experimental curves  $p \times n$  for several gases with very good precision [36]. This is a strong physical argument in favor of the set  $\{p(n, T), \rho(n, T)\}$  and indicates that the associated series are well-defined.

There is another reasoning suggesting the convergence of the series in  $n$ : the virial coefficients  $a_l$  and  $c_l$  are evaluated from a subclass of connected diagrams, the *irreducible diagrams* [10]. In the classical case, the irreducible diagrams are easily detectable, since they are *multipy connected*, i.e., each particle connects to at least other two. For example, coefficient  $b_3$ , given by (28c), is constructed from all the four diagrams shown in the right of Figure 2. The coefficients  $a_3$  and  $c_3$ , however, depend solely on the last diagram of Figure 2. The first can be written as

$$a_3 = -\frac{1}{3g^2} \iint f_{12} f_{13} f_{23} d^3 r_{12} d^3 r_{13} . \quad (37)$$

[Both in (36) and (37) we are supposing that the potential depends only on the interparticle distance,  $V(\vec{r}_i, \vec{r}_j) = V(|\vec{r}_i - \vec{r}_j|)$ .] This restriction on the diagrams for the non-relativistic classical system favors the virial expansion against the series in the fugacity. It is to be expected that convergence is better achieved by the virial expansion.

In view of these arguments, we choose the set composed by  $p(n, T)$  and  $\rho(n, T)$  as the most convenient for computing interactions in the pre-nucleosynthesis cosmological universe. Nevertheless, even this set presents a validity limit. The perturbative methods do not apply to dense systems (with high values of  $n$ ) or


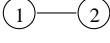
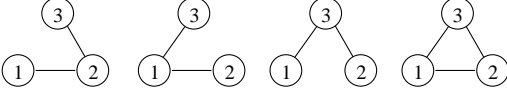
$b_1$	$b_2$	$b_3$
		

Figure 2: Classical diagrams representing the first three cluster integrals. Each line corresponds to a Mayer function and each ball, to a particle.

to systems submitted to long-range interactions. In such cases the EOS must be found by other methods.

In next section, we will construct a toy-model in order to exhibit some possible effects in the evolution of the universe when interaction between the source components is taken into account. We shall also argue in favor of the convergence of the series  $p(n, T)$  in this particular example, comparing with the behavior of the expansion  $p(z, T)$ .

## 4 Example of the interaction influence

In our toy-model the primeval interacting fluid is constituted only by photons and nucleons coming from pair production. The photons  $\gamma$  will be treated in the usual manner as ultra-relativistic ideal bosons, and the nucleons  $N$  will be considered non-relativistic interacting classical particles.

The interaction processes taken into account are:

1. Creation and annihilation of the nucleons  $N$  in the thermal bath,  $\gamma + \gamma \leftrightarrow N + \bar{N}$ . This reaction generates the mean numerical density of nucleons  $n_N$  and anti-nucleons  $n_{\bar{N}}$ . In a purely classical context, this process does not take place. Here, it serves only as a source of the interacting particles, which are treated classically as soon as they come into existence.
2. Nucleons affecting nucleons through a (very simplified) nuclear potential.

The electromagnetic interaction cancels out globally since the numerical density of nucleons are identical to that of anti-nucleons  $\bar{N}$  (Debye scenery). Weak interaction is several order of magnitude less effective than the strong interaction and it is consequently neglected.

For simplicity, we admit that the interactions  $NN$ ,  $N\bar{N}$  and  $\bar{N}\bar{N}$  are described by the same nuclear potential (charge independence of the strong interaction), and that the internal degrees of freedom come from spin and isospin. Therefore, a hadronic part of the cosmic fluid is composed by particles with mass  $m_N = m_{\bar{N}} = 938.26$  MeV (the proton rest-mass) and degeneracy  $g_N = g_{\bar{N}} = 4$ .

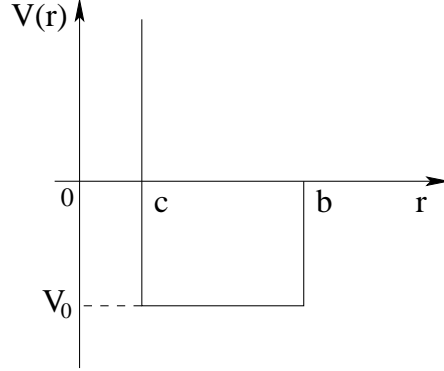


Figure 3: Potential modeling the nuclear interaction.

The nuclear interaction shall be modeled by a square-well combined with a hard-core potential, as in Figure 3.

The choice of that nuclear potential is justified by two facts: *(i)* it presents a behavior similar to those exhibited by some successful phenomenological nuclear potentials, as those described in Refs. [37, 38]; and *(ii)* with this simplified potential, it is possible to calculate analytically the cluster integrals to the third order.

The parameters of the square-well hard-core potential are obtained from (a) the deuteron binding energy, the proton and deuteron mean-squared radius – which fix the well's width  $(b - c) = 1.3$  fm and its depth,  $V_0 = 75.6$  MeV – and (b) from nucleons high-energy scattering data – setting the extension of the hard-core  $c = 0.4$  fm (cf. Ref. [39]).

The pressure and the energy density for the PNU are then written in the form:

$$p(kT, n_N) = p_\gamma(kT) + p_N(kT, n_N) , \quad (38a)$$

$$\rho(kT, n_N) = \rho_\gamma(kT) + \rho_N(kT, n_N) , \quad (38b)$$

where  $p_N$  and  $\rho_N$  are given by (33a) and (33b) respectively, and [10]

$$p_\gamma = \frac{\rho_\gamma}{3} ; \quad \rho_\gamma(kT) = \frac{\pi^2 (kT)^4}{15} . \quad (39)$$

Truncating the nucleons EOS in the third order (approximation valid for  $n_N$  not too large), explicit expressions for  $p(T, n_N)$  and  $\rho(T, n_N)$  result:

$$p(kT, n_N) \simeq \frac{\pi^2 (kT)^4}{45} + (kT) [n_N + a_2 n_N^2 + a_3 n_N^3] , \quad (40a)$$

$$\rho(kT, n_N) \simeq \frac{\pi^2 (kT)^4}{15} + (m_N + \frac{3}{2} kT) n_N + (kT)^2 [c_2 n_N^2 + c_3 n_N^3] , \quad (40b)$$

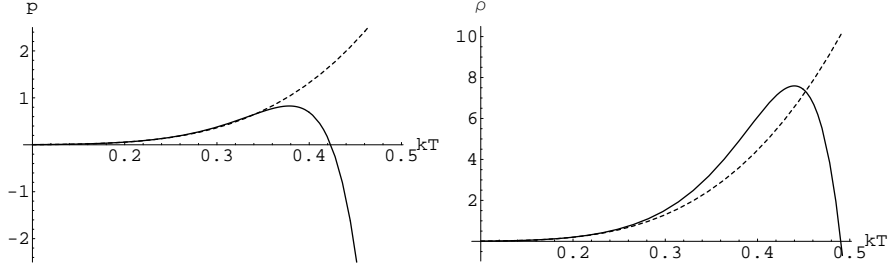


Figure 4: Graphics of the pressure  $p(kT)$  and energy density  $\rho(kT)$ , both measured in  $\text{GeV}/\text{fm}^3$ , as functions of  $kT$  given in  $\text{GeV}$ . Full lines represent  $p$  and  $\rho$  of the proposed model (with interaction). For sake of comparison, the dotted curves show  $p$  and  $\rho$  in the ideal case (without interaction).

with

$$n_N(kT) \simeq g_N \frac{e^{-\beta m_N}}{\lambda_N^3} + 2b_2 + 3b_3 . \quad (41)$$

Recall that  $\mu_N \simeq 0$  in the pre-nucleosynthesis cosmological period and, in consequence,  $z_N = 1$ . Coefficients  $a_2$  and  $a_3$  are found analytically with the help of (36) and (37), while  $b_2$ ,  $b_3$  and  $c_2$ ,  $c_3$  are derived directly from (34) and (35). Performing all calculations and inserting  $n_N(kT)$  in (40a) and (40b), we finally obtain the EOS  $p(kT)$  and  $\rho(kT)$  for the PNU. The result is presented in a graphic form – see Figure 4. For  $kT \lesssim 0.3 \text{ GeV}$ , the proposed model is qualitatively identical to the ideal case. This is an expected feature since, until  $0.3 \text{ GeV}$ , the numerical density  $n_N(kT)$  is too small to cause any relevant interaction effect. From this energy value on, the deviation from the ideal case emerges, first in the curve for  $\rho$  and then in the plot of  $p$ . As energy increases, the two functions tend to decrease and, eventually,  $p$  and  $\rho$  become negative. This peculiar characteristic is due to the action of the interaction terms in (40a) and (40b). They dominate at high-energies: the attractive part of the nuclear potential (square-well) make  $a_2$  and  $c_2$  the most important terms of the truncated expansion.

The effect of the nuclear interaction in the PNU is better seen by observing the equation of state in its modern-cosmology familiar form,

$$w(kT) \equiv \frac{p(kT)}{\rho(kT)} . \quad (42)$$

The behavior of  $w(kT)$  is shown in Figure 5. For  $kT < 0.1 \text{ GeV}$  the function  $w(kT) \simeq 1/3$ , implying that the EOS is that typical of a radiation-dominated universe. The ideal terms reduce slightly this value of  $w$  as the energy increases. But, for energy values greater than  $0.3 \text{ GeV}$ , the interaction processes become relevant, reducing abruptly the value of  $w(kT)$  and deviating its behavior from the ideal case.

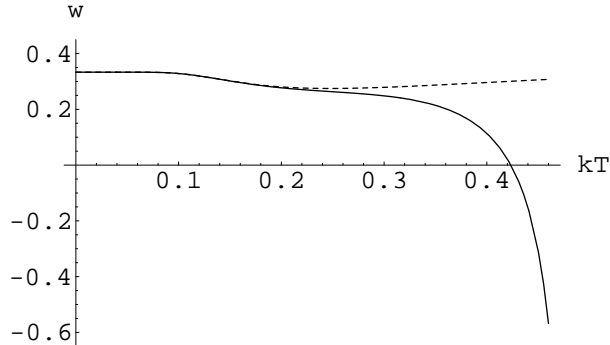


Figure 5: Parametric equation  $w(kT)$  as a function of  $kT$  (in GeV). The full curve represents the proposed model (with interaction). The dotted line corresponds to the ideal (without interaction) cosmological EOS.

The dynamical (interaction) terms modify the cosmological EOS and, consequently, change the form of the primordial expansion. In particular, our toy-model produces an accelerated expansion which naturally evolves to a decelerated radiation-like expansion, which is necessary for the nucleosynthesis. Nevertheless, we would not dare to say that this simplistic model represents realistically the PNU dynamics. This model is apt only to show the importance of considering interactions within the cosmic fluid as an important factor in the determination of the early universe's evolution.

Another argument favoring the better convergence of the pressure series  $p(kT, n_N)$  (40a) in comparison to that for the energy-density series  $p(kT, z_N)$  (32a) follows as a by-product. It is possible to compare the second interaction terms ( $a_2$  or  $b_2$ ) with the third ones ( $a_3$  or  $b_3$ ), by defining the functions

$$F_n(kT) \equiv 1 - \left| \frac{a_3 n_N^3}{a_2 n_N^2} \right| ; \quad F_z(kT) \equiv 1 - \left| \frac{b_3 z_N^3}{b_2 z_N^2} \right| . \quad (43)$$

The more are  $F_n(kT)$  and  $F_z(kT)$  close to 1, the more the third interactions terms are irrelevant compared to the second terms, and the larger is the possibility of convergence of  $p(kT, n_N)$  and  $p(kT, z_N)$ . Notice, however, that this analysis does not prove convergence: it uses only the first terms of the series.  $F_n(kT)$  and  $F_z(kT)$  only help us to find an argument favoring the good behavior of the EOS. Graphics of  $F_n$  and  $F_z$  as functions of  $kT$  are showed in Figure 6. The conclusion is that the series in terms of  $n_N$  – namely  $p(kT, n_N)$  – has a better chance to converge than the series in  $z_N$ . This corroborates the choice of section 3.1, where we have chosen the EOS set given in terms of the numerical density as the suitable EOS for the pre-nucleosynthesis cosmological universe.



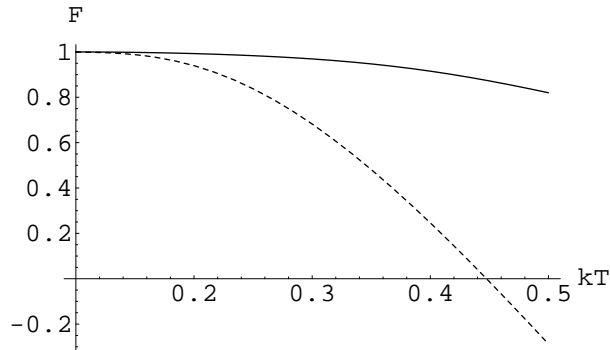


Figure 6: The functions  $F_n(kT)$  (full line) and  $F_z(kT)$  (dashed line).  $kT$  is measured in GeV.

## 5 Final remarks

This work highlights the absence of dynamical terms as direct source of curvature in the usual cosmological models. The detailed examination of this fact is performed in section 2. The analysis is trivial for the equilibrium approach, but, even in the non-equilibrium case, no mention is made of the possible relevance of interacting components for the universe evolution. Some exceptions are Refs. [9, 40, 41].

A method for including these dynamical effects in the pre-nucleosynthesis universe has been proposed, through the corrections of the EOS based in the cluster expansion technique of Statistical Mechanics. The approach is rather general and allows the treatment of the cosmic fluid as a classical or quantum system (relativistic or not). In principle, the method can be applied to other periods of cosmic history, provided the conditions for thermodynamical equilibrium and series convergence are respected.<sup>7</sup>

A toy-model has been presented which illustrates the deep consequences which interaction between constituents can have for cosmic evolution. Even if overmuch simplified, the model points the way toward more realistic approaches, based on fundamental physics. More accurate models for the pre-nucleosynthesis universe would include other particles than just photons and nucleon-anti-nucleons pairs: at least pions, kaons, electrons and neutrinos should be included. Besides, the nuclear interaction should be treated in a more complete manner than just a square-well hard-core potential. More realistic EOS are found in studies [42, 43, 44] concerning interacting hadrons.

We cannot affirm categorically that the inclusion of dynamical terms can describe properly the primordial acceleration of the universe (as suggested by

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<sup>7</sup> Such conditions are required only for the interacting part of the cosmic fluid.

our toy-model), but the results presented here are indicative that these terms should not be simply ignored, as usually done. Maybe the inflationary era and the present-day acceleration are just consequences of neglected interaction terms.

## A Appendix: Cluster expansions

The ensemble approach to Statistical Physics is able to include interactions perturbatively, via the cluster expansion formalism. The method is developed in the grand canonical ensemble, whose partition function, written for a single component fluid, is:

$$\Xi(z, V, T) = \sum_{N=0}^{\infty} Q_N(V, T) z^N \quad (44)$$

with  $z$  is given by (10) and

$$Q_N(V, T) = \frac{1}{N!} \int_{\Omega} W_N^X(\vec{r}_1, \dots, \vec{r}_N) d^3 r_1 \dots d^3 r_N, \quad (45)$$

where tag  $X$  indicates the nature of the system under consideration:  $X = C$  denotes a non-relativistic classical system and  $X = Q$ , a quantum system (QS).  $W_N^X$  is the probability density for a  $N$ -particle system,

$$W_N^C(\vec{r}_1, \dots, \vec{r}_N) \equiv \left( \frac{e^{-\beta m}}{\lambda^3} \right)^N \exp \left[ -\beta \sum_{i < j=1}^N V(\vec{r}_i, \vec{r}_j) \right], \quad (46a)$$

$$W_N^Q(\vec{r}_1, \dots, \vec{r}_N) \equiv N! \langle \vec{r}_1, \dots, \vec{r}_N | \hat{A} e^{-\beta \hat{H}_N} | \vec{r}_1, \dots, \vec{r}_N \rangle, \quad (46b)$$

where  $V(\vec{r}_i, \vec{r}_j)$  is the potential between the particles  $i$  and  $j$ ;  $\hat{A}$  is the symmetrization operator and  $\hat{H}_N$  the  $N$ -particle Hamiltonian operator.

The thermodynamical quantities are associated to the grand canonical potential  $\Omega$  which, just like the partition function, can be expressed as a series in terms of the fugacity  $z$ ,

$$\Omega(z, T) = \frac{1}{V} \ln \Xi(z, V, T) \equiv \sum_{N=1}^{\infty} b_N z^N, \quad (47)$$

with

$$b_N = \frac{g}{N!V} \int_{\Omega} U_N^X(\vec{r}_1, \dots, \vec{r}_N) d^3 r_1 \dots d^3 r_N. \quad (48)$$

The  $U_N^X$  are the *Ursell functions* and  $g$  counts the degeneracy coming from internal degrees of freedom.<sup>8</sup> The coefficients  $b_N$  are the *cluster integrals*.

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<sup>8</sup> How to count the degenerate states depends critically on the type of system. For instance, when dealing with a quantum system, the counting must respect the symmetry condition of the complete wave-function.

Using Eqs.(44), (45), (47) and (48) and expanding  $\ln \Xi$  in terms of the fugacity, one writes  $U_N^X$  in terms of  $W_N^X$ . Examples:

$$\begin{aligned} U_1^X(\vec{r}_1) &= W_1^X(\vec{r}_1) , \\ U_2^X(\vec{r}_1, \vec{r}_2) &= W_2^X(\vec{r}_1, \vec{r}_2) - W_1^X(\vec{r}_1)W_1^X(\vec{r}_2) , \\ U_3^X(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \\ &W_3^X(\vec{r}_1, \vec{r}_2, \vec{r}_3) - 3W_2^X(\vec{r}_1, \vec{r}_2)W_1^X(\vec{r}_3) + 2W_1^X(\vec{r}_1)W_1^X(\vec{r}_2)W_1^X(\vec{r}_3). \end{aligned} \quad (49)$$

In principle, all the possible states of a statistical system can be decomposed in generic diagrams accounting for the several correlation processes (interactions or quantum exchange effects). The  $Q_N$ , determined from the probability density functions, are obtained as the sum of all distinct diagrams of the  $N$  particles. The cluster integrals, determined from the Ursell functions, are obtained as the sum of all connected diagrams,

$$b_N = \frac{g}{V} \sum (\text{connected diagrams of } N \text{ particles}) . \quad (50)$$

In a  $N$ -particle connected diagram all the  $N$  particles are linked directly or indirectly. The links include all kinds of correlation.

The arrangement of the several distinct diagrams in connected diagrams is complicated and it will not be carried out here. The interested reader may consult Refs. [10] for classical systems, and [34] for the quantum case.

## A.1 Non-relativistic classical system

According to (46a):

$$U_1^C(\vec{r}_1) = W_1^C(\vec{r}_1) = \frac{e^{-\beta m}}{\lambda^3} , \quad (51)$$

$$U_2^C(\vec{r}_1, \vec{r}_2) = \left( \frac{e^{-\beta m}}{\lambda^3} \right)^2 f(\vec{r}_1, \vec{r}_2) , \quad (52)$$

where

$$f(\vec{r}_i, \vec{r}_j) \equiv e^{-\beta V(\vec{r}_i, \vec{r}_j)} - 1 \quad (53)$$

is the *Mayer function*.

Also from (46a), it is possible to show that the non-relativistic classical cluster functions can be decomposed in products of Mayer functions [36]. For instance:

$$U_3^C(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \left( \frac{e^{-\beta m}}{\lambda^3} \right)^3 \left( e^{-\beta V(\vec{r}_1, \vec{r}_2)} e^{-\beta V(\vec{r}_1, \vec{r}_3)} e^{-\beta V(\vec{r}_2, \vec{r}_3)} - 3e^{-\beta V(\vec{r}_1, \vec{r}_2)} + 2 \right) .$$

$$\begin{aligned} U_3^C &= \left( \frac{e^{-\beta m}}{\lambda^3} \right)^3 \left[ \left( e^{-\beta V(\vec{r}_1, \vec{r}_2)} - 1 \right) \left( e^{-\beta V(\vec{r}_1, \vec{r}_3)} - 1 \right) \left( e^{-\beta V(\vec{r}_2, \vec{r}_3)} - 1 \right) + \right. \\ &+ e^{-\beta V(\vec{r}_1, \vec{r}_2)} e^{-\beta V(\vec{r}_1, \vec{r}_3)} + e^{-\beta V(\vec{r}_1, \vec{r}_2)} e^{-\beta V(\vec{r}_2, \vec{r}_3)} + e^{-\beta V(\vec{r}_1, \vec{r}_3)} e^{-\beta V(\vec{r}_2, \vec{r}_3)} + \\ &\left. - 2e^{-\beta V(\vec{r}_1, \vec{r}_2)} - 2e^{-\beta V(\vec{r}_1, \vec{r}_3)} - 2e^{-\beta V(\vec{r}_1, \vec{r}_2)} + 3 \right] . \end{aligned}$$

$$U_3^C(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \left( \frac{e^{-\beta m}}{\lambda^3} \right)^3 [f(\vec{r}_1, \vec{r}_2)f(\vec{r}_1, \vec{r}_3)f(\vec{r}_2, \vec{r}_3) + f(\vec{r}_1, \vec{r}_2)f(\vec{r}_1, \vec{r}_3) + f(\vec{r}_1, \vec{r}_2)f(\vec{r}_2, \vec{r}_3) + f(\vec{r}_1, \vec{r}_3)f(\vec{r}_2, \vec{r}_3)] . \quad (54)$$

Such a decomposition can be obtained for all orders. The Mayer functions will always be the fundamental entities in the determination of the  $U_N^C(\vec{r}_1, \dots, \vec{r}_N)$ , and lead to a representation of the cluster integrals in terms of connected diagrams — the first three are shown in Figure 2.

The number of connected diagrams grows very fast as  $N$  increases: there is only one diagram if  $N = 2$ , 4 diagrams for  $N = 3$ , and 38 for  $N = 4$ .

## A.2 Relativistic quantum system

Relativistic quantum system is much more complicated than the non-relativistic classical system by several reasons:

- In a relativistic system an interaction cannot be represented by a potential. In fact, the determination of an interaction Hamiltonian  $H' \equiv H - H_0$ , the total Hamiltonian  $H$  minus the free Hamiltonian  $H_0$ , is a difficult task.
- There are quantum correlation effects which are mixed to the dynamical (interaction) terms.
- There are discrete bound states.

Despite these difficulties, there exists a general formalism giving the coefficients  $b_N$  in terms of the S-matrix [34].

Analogously to (46b), we define the operator  $\hat{U}_N$  such as

$$U_N^Q(\vec{r}_1, \dots, \vec{r}_N) \equiv N! \langle \vec{r}_1, \dots, \vec{r}_N | \hat{U}_N | \vec{r}_1, \dots, \vec{r}_N \rangle . \quad (55)$$

Then, using (48),

$$b_N = \frac{g}{V} \int_{\Omega} \langle \vec{r}_1, \dots, \vec{r}_N | \hat{U}_N | \vec{r}_1, \dots, \vec{r}_N \rangle d^3 r_1 \dots d^3 r_N , \quad (56)$$

or through a Fourier transform,

$$b_N = \frac{g}{V} \int_{\Omega_k} \langle \vec{k}_1, \dots, \vec{k}_N | \hat{U}_N | \vec{k}_1, \dots, \vec{k}_N \rangle d^3 k_1 \dots d^3 k_N . \quad (57)$$

The general form of the  $b_N$  is

$$b_N = \frac{g}{V} \text{Tr} \hat{U}_N . \quad (58)$$

Equation (56) is the realization of this trace in coordinate representation, and (57) is its realization in the momentum representation.

In order to obtain the  $b_N$  in terms of the S-matrix, it is necessary to separate the statistical effects from the dynamical effects, because only the latter influence the S-matrix. The traditional method to accomplish this is to construct

$$b_N^{(0)} = \frac{g}{V} \text{Tr} \hat{U}_N^{(0)} , \quad (59)$$

where index (0) denotes a free system (without interaction).  $\hat{U}_N^{(0)}$  is related to a free system, it is responsible only for statistical contribution. Therefore, it is sufficient to subtract (59) from (58) to retain the pure dynamical terms. This gives

$$b_N - b_N^{(0)} = \frac{g}{V} \text{Tr} \left( \hat{U}_N - \hat{U}_N^{(0)} \right) . \quad (60)$$

Thus, the grand canonical potential (47) is:

$$\Omega - \Omega_0 = \sum_{N=1}^{\infty} \left( b_N - b_N^{(0)} \right) z^N , \quad (61)$$

where  $\Omega_0$  is the ideal grand canonical potential (9). Dashen, Ma and Bernstein [34] showed that (60) can be written in terms of the S-matrix as:

$$\text{Tr} \left( \hat{U}_N - \hat{U}_N^{(0)} \right) = \int \frac{e^{-\beta E}}{4\pi i} \text{Tr} \left( \hat{A} \cdot \hat{S}^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} \hat{S} \right)_{c_N} dE , \quad (62)$$

where  $\hat{A}$  is the symmetrization operator,  $\hat{S}$  is the on-shell S-matrix operator [35], and  $c_N$  indicate that only the connected  $N$ -particle diagrams are to be considered.

Taking (62) into (60) leads to the general expression (30) for the cluster integrals  $b_N$ . That form of  $b_N$  includes not only the scattering states, but also the (bound state) composed particles (for more details see [34]).

Let us exhibit the explicit expressions of (62) for one-particle and two-particle systems.

1.  $N = 1$ :

$$\text{Tr} \left( \hat{U}_1 - \hat{U}_1^{(0)} \right) = \text{Tr} \left( \hat{W}_1 \right) - \text{Tr} \left( \hat{W}_1^{(0)} \right) = \text{Tr} \hat{A} e^{-\beta \hat{H}_1} - \text{Tr} \hat{A} e^{-\beta \hat{H}_1^{(0)}} = 0 . \quad (63)$$

The last equality results from  $\hat{H}_1 = \hat{H}_1^{(0)}$  (free particle).

2.  $N = 2$ :

$$\begin{aligned} \text{Tr} \left( \hat{U}_2 - \hat{U}_2^{(0)} \right) &= \text{Tr} \hat{W}_2 - \text{Tr} \left( \hat{W}_1 \hat{W}_1 \right) - \left( \text{Tr} \hat{W}_2^{(0)} - \text{Tr} \left( \hat{W}_1^{(0)} \hat{W}_1^{(0)} \right) \right) . \\ \text{Tr} \left( \hat{U}_2 - \hat{U}_2^{(0)} \right) &= \text{Tr} \left( \hat{W}_2 \right) - \text{Tr} \left( \hat{W}_2^{(0)} \right) . \end{aligned}$$

$$\begin{aligned}
Tr \left( \hat{U}_2 - \hat{U}_2^{(0)} \right) &= Tr \hat{A} e^{-\beta \hat{H}_2} - Tr \hat{A} e^{-\beta \hat{H}_2^{(0)}} \\
&= \int \frac{e^{-\beta E}}{4\pi i} Tr \left( \hat{A} \hat{S}_2^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} \hat{S}_2 \right) dE, \quad (65)
\end{aligned}$$

where  $\hat{H}_2$  is the two-particle Hamiltonian operator and  $\hat{S}_2$  is the S-matrix operator associated to  $\hat{H}_2$ . It is worth to remark that (65) can be rewritten in terms of measurable *phase-shifts*, if we choose the angular momentum representation.

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